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CLIFFORD MULTIPLICATION AND f-STRUCTURES

CLARK JEFFRIES

1. Introduction

M. F. Atiyah [1], [2] has neatly applied Clifford multiplication of exterior forms on (smooth, compact) Riemannian manifolds to certain reduction problems of the structure groups of tangent bundles, and considered Clifford multiplication by orientation forms associated with global splittings of tangent bundles into subbundles, that is, plane fields.

We suppose an m-dimensional manifold M admits a (1,1)-tensor solution f of $f^3 + f = 0$ with (constant) rank $2l > 0$, that is, an f-structure. One may choose a Riemannian structure $\mathscr{G}$ for M so that f is skew. Thus the tangent bundle $T(M)$ of M splits globally as the sum of ker f and the orthogonal complement ker $f^{\perp}$, on which f induces an almost complex structure. Associated with f and $\mathscr{G}$ is a 2-form ω. The purpose of this paper is to study Clifford multiplication by ω and the orientation form $(\bigwedge \omega)^l$ of the plane field ker $f^{\perp}$.

The existence of an f-structure is, of course, equivalent to the reduction of the structure group of $T(M)$ from $\mathscr{O}(m)$ to $\mathscr{O}(m-2l) \times \mathscr{U}(l)$. The literature devoted to f-structures and related topics is extensive, beginning, it seems, with K. Yano [4].

2. Algebraic considerations

First we review Clifford multiplication. Clifford multiplication of cross sections of the exterior algebra Λ of M depends upon the choice of Riemannian structure $\mathscr{G}$. We consider $\mathscr{G}$ as extended throughout the tensor algebra of M. Right and left Clifford multiplications are algebra homomorphisms from cross sections of Λ to function-linear cross sections of Hom (Λ, Λ). Suppose α is a p-form and β a q-form. Define the adjoint of exterior multiplication $\bigwedge$ as follows. If $p < q$, then $\alpha \vee \beta = 0$. If $p \geq q$, then

$$\alpha \vee \beta = \sum_j \mathscr{G}(\alpha, \beta \wedge \mu_j)\mu_j ,$$

where $\{\mu_j\}$ is a local orthonormal basis of the $p-q$ floor of Λ. This extends to a global definition of $\alpha \vee \beta$. If v is a 1-form and α is a p-form, then define the Clifford product $v \cdot \alpha$ as $v \cdot \alpha = v \wedge \alpha - \alpha \vee v$. If $v_1, \cdots, v_q$ are ortho-

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normal 1-forms, then define $(v_1 \wedge \cdots \wedge v_q)\cdot\alpha = v_1\cdot(\cdots(v_q\cdot\alpha)\cdots)$. Clearly this extends to a global multiplication, Clifford multiplication, for any pair of cross sections α and β of Λ. This leads to right and left Clifford multiplications $\boldsymbol{R}$ and $\boldsymbol{L}$; that is, $\alpha\cdot\beta = \boldsymbol{R}_\beta(\alpha) = \boldsymbol{L}_\alpha(\beta)$. Clifford multiplication is associative, so $\boldsymbol{R}_\beta$ and $\boldsymbol{L}_\alpha$ always commute.

Now suppose v_1 and v_2 are local orthonormal 1-forms. Then locally,

$$\boldsymbol{R}_{v_1}^2 = -I ,$$

where here and hereafter I denotes the identity transformation in whatever context it may occur. Also, $\boldsymbol{R}_{v_1}$ is skew with respect to the natural extension of $\mathscr{G}$ to Hom(Λ, Λ). Note that $\boldsymbol{R}_{v_1}\boldsymbol{R}_{v_2} = -\boldsymbol{R}_{v_2}\boldsymbol{R}_{v_1}$ and $\boldsymbol{R}_{v_1\wedge v_2} = \boldsymbol{R}_{v_2}\boldsymbol{R}_{v_1}$. Lastly, $\boldsymbol{R}_{v_1}\colon \Lambda^{\text{even}} \to \Lambda^{\text{odd}}$ and $\boldsymbol{R}_{v_1}\colon \Lambda^{\text{odd}} \to \Lambda^{\text{even}}$.

Given a global oriented plane field on M of dimension k, we may locally express a (unit) orientation form A for the plane field as some exterior product $A = v_1 \wedge \cdots \wedge v_k$ of orthonormal 1-forms. Thus $\boldsymbol{R}_A$ is locally $\boldsymbol{R}_{v_k}\cdots\boldsymbol{R}_{v_2}\boldsymbol{R}_{v_1}$, so globally $\boldsymbol{R}_A^2 = +I$ if $k \equiv 0, 3 \bmod 4$ and $\boldsymbol{R}_A^2 = -I$ if $k \equiv 1, 2 \bmod 4$. Such operators were used extensively in [2].

Now suppose M admits an f-structure f, an adapted Riemannian structure $\mathscr{G}$, and an associated 2-form ω all as in the Introduction.

We first derive a minimal polynomial satisfied by $\boldsymbol{R}_\omega$. The two lemmas which follow may be proved easily using the following fact: Given two vector space homomorphisms which commute and are almost complex, there is a natural splitting of the vector space into two subspaces with the homomorphisms equal on one subspace and additive inverses on the other.

Lemma 1. *The sum J of $2p+1$ commuting almost complex vector space homomorphisms (not necessarily distinct) satisfies*

$$\prod_{j\ \text{odd},\, 1\le j\le 2p+1} (J^2 + j^2 I) = 0 .$$

Lemma 2. *The sum J of $2p$ commuting almost complex vector space homomorphisms satisfies*

$$\prod_{j\ \text{even},\, 0\le j\le 2p} (J^2 + j^2 I) = 0 ,$$

that is,

$$J \prod_{j\ \text{even},\, 2\le j\le 2p} (J^2 + j^2 I) = 0 .$$

Note that one may always choose an inner product so that each of the given almost complex homomorphisms is skew, and hence so that J is skew. Now since ω may be locally expressed as $v_1 \wedge v_2 + v_3 \wedge v_4 + \cdots + v_{2l-1} \wedge v_{2l}$, it follows that $\boldsymbol{R}_\omega$ may be locally expressed as the sum of l commuting almost complex structures, namely $R_\omega = R_{v_2\wedge v_1} + \cdots + R_{v_{2l}\wedge v_{2l-1}}$. In view of Lemmas 1 and 2, we have

Lemma 3.

$$\prod_{j\ \mathrm{odd},1\le j\le l} (R_\omega^2 + j^2 I) = 0 \qquad \textit{for}\ \text{odd}\ l\,,$$

$$R_\omega \prod_{j\ \mathrm{even},2\le j\le l} (R_\omega^2 + j^2 I) = 0 \qquad \textit{for}\ \text{even}\ l\,.$$

Furthermore, these are minimal polynomials for R_ω. For if we apply R_ω repeatedly to the constant 0-form 1, then we have $R_\omega^s(1) = (\bigwedge\omega)^s +$ (forms of degree less than $2s$).

Now let J be the sum, and K be the product of $2p$ commuting almost complex vector space homomorphisms. Thus $K^2 = I$. Using an inner product with respect to which each of the almost complex homomorphisms is skew, consider the orthogonal $+1$ and -1 eigenspaces of K. Since J and K commute, we may write $J = J_+ + J_-$ where J_+ and J_- denote the restrictions of J to the eigenspaces. We have

Lemma 4. *J_+ restricted to the $+1$ eigenspace of K satisfies*

$$J_+ \prod_{j\ \mathrm{even},2\le j\le p} (J_+^2 + (2j)^2 I) = 0\,,$$

and J_- restricted to the -1 eigenspace of K satisfies

$$\prod_{j\ \mathrm{odd},1\le j\le p} (J_-^2 + (2j)^2 I) = 0\,.$$

Again the proof uses only elementary linear algebra and is omitted.

It is clear that the relations in Lemma 4 hold globally for R_ω and R_A, provided rank $f = 2l \equiv 0 \bmod 4$.

3. Analytic results

We assume henceforth that the dimension of M is congruent to 1 mod 4, and that M is compact and orientable. Let B denote a unit orientation form for M. Let d and d^* denote the usual exterior and coexterior derivatives. It may be shown that $L_B^2 = -I$ and that L_B commutes with $d + d^*$ when restricted to Λ^{even}. We may form an elliptic differential operator T of degree one by setting $T = L_B(d + d^*)\colon \Lambda^{\mathrm{even}} \to \Lambda^{\mathrm{even}}$. Now the symbol of $d + d^*$ is $\sqrt{-1}L_B$. Thus, since R and L commute, $R_{v_1\wedge v_2}$ commutes with T in highest, that is, first order terms, where $v_1 \wedge v_2$ is a local unit 2-form. Similarly any real polynomial of images of such unit 2-forms commutes with T in first order terms. Recall that the real Kervaire semi-characteristic of an odd-dimensional manifold is the sum mod 2 of the even Betti numbers of the manifold. Thus (dim ker T) mod 2 $= k(M)$ the real Kervaire semi-characteristic of M. Finally, note that T is skew with respect to the usual extension of $\mathscr{G}$ to the (infinite-dimensional) real vector space Λ^{even} over (compact) M.

Next we prove

Theorem 1. *Suppose R is a cross section of* $\text{Hom}(\Lambda^{\text{even}}, \Lambda^{\text{even}})$ *which commutes with T in first order terms. If R satisfies a minimal polynomial $p(x) = x^s + \cdots + b_1 x + b_0$ with s distinct roots, real or complex, then we may choose constants a_{ij}; $i = 0, \cdots, s-1$; $j = 1, \cdots, s-1$, such that the new differential operator*

$$S = T + \sum_{i,j} a_{ij} R^i (TR^j - R^j T)$$

commutes with R. Furthermore, if the adjoint of R is a polynomial in R, then S is skew.

Proof. We note that any such S has the same first order terms as T.

Next we derive the numbers $\{a_{ij}\}$ in terms of $p(x)$. One may regard $TR^j - R^j T$ as the derivative of R^j with respect to T. Thus we will define S so that the derivative of R with respect to S is 0. We generally follow our proof of a different version of this result involving connections in vector bundles given in [3, Theorem 1].

Our first step is to complexify the real vector bundle Λ so that if $\lambda_1, \cdots, \lambda_s$ are the distinct roots of $p(x)$, then $R = \sum_{a=1,\cdots,s} \lambda_a \pi_a$ where $\{\pi_a\}$ are projections onto the eigenbundles of R. Our last step will be to take the real parts of the constants $\{a_{ij}\}$ which we derive, and this will obviously suffice.

Each π_a may be described explicitly as follows. Define new complex polynomials $p_a(x)$ by

$$p_a(x) = \prod_{b \neq a} (x - \lambda_b) .$$

Then $\pi_a = p_a(\lambda_a)^{-1} p_a(R)$. Thus $\pi_a \pi_b = \delta_{ab} \pi_a$, $\sum \pi_a = I$, and $R = \sum \lambda_a \pi_a$.

Define a new operator $\tilde{S}$ on the complexification of Λ by

$$\tilde{S} = T + \sum_{a=1}^{s} \pi_a \{T\pi_a - \pi_a T\} = \sum_{a=1}^{s} \pi_a T \pi_a .$$

Now for each π_b, $\tilde{S}\pi_b - \pi_b \tilde{S} = 0$, so $\tilde{S}R - R\tilde{S} = 0$. Note also that if the adjoint of R is a polynomial in R, that is, if the complex ajoint of each π_b is itself, then the complex adjoint of $\tilde{S}$ is $-\tilde{S}$.

Now define complex numbers c_{ij}; $i = 1, \cdots, s$; $j = 1, \cdots, s-1$, by $\pi_i = \sum_j c_{ij} R^j$. It follows that

$$\tilde{S} = T + \sum_{i=0}^{s-1} \sum_{j=1}^{s-1} a_{ij} R^i [TR^j - R^j T] ,$$

where $a_{ij} = \sum_{k=1}^{s} c_{ki} c_{kj}$.

Clearly the real part S of $\tilde{S}$ on Λ^{even} satisfies $SR - RS = 0$ and has the same first order terms as T. Also, S is skew provided the adjoint of R is a polynomial in R. q.e.d.

In the special case $p(x) = x^2 + 1$, $S = T - \frac{1}{2}\{TR - RT\} = \frac{1}{2}\{T + RTR^{-1}\}$, as originally used in [2, p. 16].

Theorem 1 simply implies that the (real finite dimensional) kernel of S admits R. We will be concerned with the case where R is skew and the case where $\tilde{p}(R)$ [1] $\neq 0$ unless p divides $\tilde{p}$, that is, the case when p remains the minimal polynomial of R in $\ker S$.

4. Applications

Among other results, Atiyah showed that if a compact, orientable, ($\equiv 1$ mod 4)-dimensional manifold admits an orientable plane field of dimension $\equiv 2$ mod 4 (or, complementarily $\equiv 3$ mod 4), then the real Kervaire semi-characteristic of the manifold vanishes. We next derive some associated results for f-structures with rank $\equiv 0$ mod 4.

Theorem 2. *Suppose M admits an f-structure of rank $4l$ with associated 2-form ω and associated cross section $\boldsymbol{R}_\omega$ of* Hom $(\Lambda^{\text{even}}, \Lambda^{\text{even}})$. *Let S be defined in terms of $\boldsymbol{R}_\omega$ and T according to Theorem* 1. *Then the dimension* mod 2 *of the kernel of $\boldsymbol{R}_\omega$ in the kernel of S is $k(M)$.*

Proof. $\boldsymbol{R}_\omega$ is skew and commutes with T in first order terms. Therefore S defined from Theorem 1 is a skew elliptic differential operator with the same first order terms as T. Using the stability of the mod 2 index as explained in [2], it follows that $\dim\ker S \equiv \dim\ker T$ mod 2. According to Lemma 2,

$$p(x) = x \prod_{j\text{ even}, 2 \leq j \leq 2l} (x^2 + j^2)$$

has the property $p(\boldsymbol{R}_\omega) = 0$. Applying $\boldsymbol{R}_\omega$ to the constant 0-form 1 show that $p(x)$ is the minimal polynomial of $\boldsymbol{R}_\omega$ in $\ker S$ as well as in Λ^{even}. Thus

$$\dim\ker \boldsymbol{R}_\omega \text{ in } \ker S \equiv \dim\ker S \equiv \dim\ker T \equiv k(M) \bmod 2. \quad \text{q.e.d.}$$

In view of Lemma 1, the analogous considerations for an f-structure of rank $4l + 2$ would lead to the vanishing of $k(M)$, a consequence already implied just by the existence of an orientable $(4l + 2)$-plane.

Now suppose M admits two f-structures e and f, both of rank $4l$ and both skew with respect to $\mathscr{G}$. Suppose $\ker e = \ker f$. Let ψ and ω denote the associated 2-forms. We will say such f-structures are orientation complementary provided $(\wedge\psi)^{2l} = -(\wedge\omega)^{2l}$ (necessarily $(\wedge\psi)^{2l} = \pm(\wedge\omega)^{2l}$).

Theorem 3. *If M admits two orientation complementary f-structures of rank $4l$, then $k(M) = 0$.*

Proof. Denote the orientation form $(\wedge\psi)^{2l}$ for the $4l$-plane $\ker e$ by A.

Consider the cross sections R_ψ, R_ω, and R_A of Hom $(\Lambda^{\text{even}}, \Lambda^{\text{even}})$. Define a new cross section R by

$$R = \tfrac{1}{2}(I - R_A)R_\psi + \tfrac{1}{2}(I + R_A)R_\omega \; .$$

It follows from Lemma 4 that on the -1 eigenvalue of R_A in Λ^{even}, R_ψ has minimal polynomial

$$p(x) = \prod_{j \text{ odd}, 1 \le j \le l} (x^2 + (2j)^2) \; .$$

Similarly, $p(x)$ is the minimal polynomial of R_ω on the $+1$ eigenbundle of R_A. Thus the minimal polynomial of R is $p(x)$. Now R_ψ, R_ω and R_A, and hence R, all commute with T in first order terms. Since R_A is symmetric and R_ψ and R_ω are skew, R is skew. Applying Theorem 1 leads to a skew elliptic operator S, which commutes with R and has the same first order terms as T. In view of the minimal polynomial of R, dim ker S is even. Thus $k(M) = 0$. q.e.d.

Note that if e and f are two orientation complementary f-structures of rank 4, then the associated 4-plane necessarily splits as the sum of two 2-planes with $e = f$ on one and $e = -f$ on the other. Since the existence of an orientable 2-plane implies $k(M) = 0$, Theorem 3 is of no interest for orientation complementary f-structures of rank 4.

On the other hand, spheres S^{4l+3} of dimension greater than seven and congruent to 3 mod 4 admit triplets of f-structures with rank $4l$ and equal kernels. Since $k(S^{4l+3}) = 1$, Theorem 3 implies no two such f-structures could be orientation complementary.

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Carleton University, Ottawa